Section de mathématiques

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Functional analysis 2
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Sheet 5

Exercises for 'Functional Analysis 2' [MATH-404]

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Ex 5.0 (A non-trivial result from linear algebra [optional]) Let X be a vector space and $f_1, \dots, f_n, f : X \to \mathbb{R}$ be linear functionals. Show that the following properties are equivalent :

a) there exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $f = \sum_{i=1}^n \lambda_i f_i$.

b) $\bigcap_{i=1}^{n} \mathrm{Ker}(f_i) \subset \mathrm{Ker}(f)$.

Hint: For the nontrivial implication consider the mapping $\Phi((f_1(x), \dots, f_n(x))) := f(x)$. Show that it is well-defined on $(f_1, \dots, f_n)(X) \subset \mathbb{R}^n$ and extend it.

Ex 5.1 (On the weak*-topology on a TVS)

Let X be a TVS with dual space X' and denote the weak*-topology on X' by τ' (cf. Definition 1.32).

a) Show that (X', τ') is a locally convex topological vector space.

b) Show that a sequence $(x'_n)_{n\in\mathbb{N}}$ converges to x' in (X', τ') if and only if $x'_n(x) \to x'(x)$ for all $x \in X$.

c)* Let X be a locally convex topological vector space. Show that (X', τ') is metrizable if and only if X has a countable algebraic base. [1]

Hint: Recall Theorem 1.17 and use Exercise 5.0 for certain functionals on X'.

Ex 5.2 (On continuity of differentiation and multiplication in $C^\infty(\Omega)$ and $\mathcal{D}_K$)

Let $\Omega \subset \mathbb{R}^d$ be open and $K \subset \mathbb{R}^d$ be compact. Show that

a) for any $\alpha \in \mathbb{N}_0^d$, the mapping $D^\alpha \colon \varphi \mapsto D^\alpha\varphi$ is a continuous linear operator in both $C^\infty(\Omega)$ and $\mathcal{D}_K$;

b) for any $f \in C^\infty(\mathbb{R}^d)$, the mapping $M_f \colon \varphi \mapsto f\varphi$ is a continuous linear operator in both $C^\infty(\Omega)$ and $\mathcal{D}_K$.

Ex 5.3 (An incomplete locally convex topology on test functions*)

Let $\Omega \subset \mathbb{R}^d$ be open and consider the set of test functions $\mathcal{D}(\Omega)$ equipped with the family of norms

$$\|\varphi\|_n = \max\{|D^\alpha\varphi(x)| : \ |\alpha| \le n, \ x \in \Omega\}, \quad n \in \mathbb{N}. \qquad (\star)$$

1. A comment on Exercise 5.1c) : its consequences shouldn't be misunderstood. If for instance (X, d) is a metric topological vector space with a countable algebraic base, then necessarily is must have a finite algebraic base! This is a consequence of Baire's category theorem. In particular, Exercise 5.1c) should be seen as a statement in the negative : if X is infinite dimensional, then typically one should not expect (X', τ') to be metrizable.

a) Assume that $\Omega = \mathbb{R}$. Pick $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}(\varphi) = [0,1]$ and $\varphi > 0$ on $(0,1)$. Define

$$\psi_m(x) = \sum_{i=1}^{m} \frac{1}{i}\varphi(x-i).$$

Show that $(\psi_m)_m$ is a Cauchy sequence in $\mathcal{D}(\mathbb{R})$, but the pointwise limit $\psi_\infty = \lim \psi_m$ does not have compact support, hence it is not in $\mathcal{D}(\mathbb{R})$.

b) Show that for any open set $\Omega \subset \mathbb{R}^d$, the space $\mathcal{D}(\Omega)$ with the suggested topology is not complete.

Hint: For $\Omega \neq \mathbb{R}^d$, consider disjoint balls accumulating at the boundary and construct a sequence of functions appropriately modifying the example in item a).

Ex 5.4 (Heine–Borel and normability)

Recall that a TVS X has the *Heine-Borel property* if every bounded and closed subset of X is compact.

a) Prove that every normed vector space $(X, \|\cdot\|)$ that has the Heine–Borel property is finite dimensional.

Hint: Recall Exercise 4.4.

b) Deduce that if $K \subset \mathbb{R}^d$ is compact with non-empty interior, the space $\mathcal{D}_K$ is not normable.